ON THE NOTION OF SURFACE AREA FOR SOLID OF REVOLUTION
(О понятии площади поверхности тел вращения)

N. Gorchev,
«St Cyril and St Methodius» University of Veliko Turnovo,
BULGARIA

Розроблений методично доцільний і систематизований підхід для введення поняття площин бічної поверхні тіл обертання. Для цієї мети використана міра Пеано-Жордано множини і принцип Дедекінда. Мета розробки – допомогти студентам – майбутнім учителям ознайомитися з цим поняттям.

Ключові слова: площа поверхні, циліндр, конус, куля, опуклий многогранник.

1. Introduction. Rigorous study of sets of points, for which one can define axiomatically the notion of measure leads to the measurable, according to Peano-Jordan, sets of points. They are a minimal class of sets of points for which the measure is uniquely defined. In modern science Lebesgue measure is used more often than Peano-Jordan measure. However the later will not lose its methodological value because of its simplicity and naturality. This is why we will consider Peano-Jordan measure as a base of our study. In particular we will consider surface area of a cylinder, cone and sphere.

In the educational-methodological literature for secondary schools (not only in Bulgaria) two approaches for study of surface area are used:

“Unfolding” method;
Minkowski’s method.

From methodological point of view these two methods are appropriate for high school students. However, when educating mathematics teachers, this notion should be defined with necessary rigor. They need to be aware of the eventual theoretical compromises connected with the use of these two methods. For example, not every surface admits “unfolding”: From Differential Geometry is known that the surfaces that preserve their area after unfolding are the ones with zero Gaussian curvature. All of these surfaces are isometric to the plane. Cylinders and cones are examples of such surfaces, but the sphere is not. Suppose there was an isometric map $\varphi$, from the sphere onto the plane. Then $\varphi$ would map a small spherical “hat” with center $A$ onto a subset of the plane and every curve with length $l$ would be mapped to a curve in the plane of the same length $l$. The boundary $b$ of the “hat” would be mapped to a circle with center $\varphi(A)$. The radius of that circle would be the length of the shortest arc, connecting $A$ with $b$. But clearly the length of such an arc would be bigger than the length of $b$, divided by $2\pi$, which is impossible.

Therefore one approach for motivating the first method consists of defining an isometric map from the surface into the plane.

The essence of Minkowski’s method is to use the notion of derivative implicitly. One has to define $\varepsilon$-neighborhood of a geometric body and a distance from a point to a geometric body. However the surface area, defined in this way, does not satisfy an important property that measure should have: additivity. To see this consider the following example: Let $S$ be a bounded plane figure in a plane with Cartesian coordinate system $Oxy$, let $S_1$ be the set of points in $S$ with rational coordinates and $S_2$ the ones with irrational coordinates. Then $\mu(S) = \mu(S_1) = \mu(S_2)$, where $\mu$ is Minkowski’s measure, and therefore $\mu(S = S_1 \cup S_2) < \mu(S_1) + \mu(S_2)$, which shows that $\mu$ is not additive ($S_1 \cap S_2 = \emptyset$).

2. Formulation of the Problem. The introduction of notion of surface area of a rotational surface in one of these two ways in the mathematical courses in our high schools is methodologically justified. But when educating students to become high school teachers another approach is needed for introduction of these notions with the necessary rigor.

Such approach is undertaken in the mathematical analysis courses.

In this note we use an approach based on Dedekind completeness principle [2] to study the surface area of cylinder, cone and sphere, suitable for high school.
Our argumentation will be based on polyhedrons (for which their surface area is known). A natural way to define surface area of any convex surface is by approximating it by surface areas of polyhedrons, inscribed in, or circumscribed around that surface. By an inscribed prism in a cylinder we mean a prism with all side edges, contained in the side boundary of the cylinder, and parallel to the axis of the cylinder, having bases, contained in the bases of the cylinder. By a circumscribed prism around a cylinder we mean a prism, each side face of which is tangent to the side boundary of the cone and parallel to the axis of the cylinder, having bases, containing the bases of the cylinder. By an inscribed pyramid in a cone we mean a pyramid with the same vertex as the cone, having bases, containing the bases of the cylinder. By a circumscribed pyramid around a cone we mean a pyramid with the same vertex as the cone, the base of which is a polygon inscribed in the base of the cone. By a circumscribed pyramid around a cone we mean a pyramid with the same vertex as the cone, the base of which is a polygon, circumscribed around the base of the cone.

Schwarz’s example [3] shows that even for relatively simple surfaces such as the cylinder, the sequence of the surface areas of the inscribed polyhedrons is not convergent, although the right circular cylinder has a well defined surface area.

Moreover consider a sequence of inscribed bodies in a right circular cone, each consisting of right circular cylinders with equal heights, and with heights going to zero. The limit of the surface areas of these bodies will be bigger than the surface area of the cone, since in a right angled triangle the sum of the catheti is bigger than the hypotenuse.

These examples show that the definition of surface area as a sequence of surface areas is not completely analogous to the corresponding definition of arc length of a curve.

3. Surface area of a right circular cylinder. We recall the following lemma due to Hadamard [1]:

Lemma: The surface area of a convex polyhedron is not bigger than the surface area of any convex polyhedron, encompassing the first. Equality holds if and only if both polyhedra coincide.

We say that a bounded convex surface $P$ encompasses a bounded convex surface $Q$ if $Q$ is contained in the closure of the bounded subset of the space with boundary $P$. We will write $P \supset Q$.

Theorem 1: For each right circular cylinder there exists a unique number, bigger than the surface areas of the inscribed right prisms and less than the surface areas of the circumscribed right prisms.

Proof: Take a right circular cylinder $C$ with bases $k(O,R)$, $k_1(O_1,R)$ and height $h$.

We inscribe a right regular triangular prism $P_1$, double its edges of the base and obtain $P_2$, double the edges of the base of $P_2$, and obtain $P_3$ and so on. In each step the vertices of the base of one prism are among the vertices of the subsequent ones. In this way we obtain a sequence $P_1, P_2, \ldots, P_n, \ldots$ of inscribed right regular prisms. Let $\mu(P_1), \mu(P_2), \ldots, \mu(P_n), \ldots$ be the sequence of their surface areas. We construct a sequence of circumscribed prisms $Q_1, Q_2, \ldots, Q_n, \ldots$ by taking the tangent planes at each edge with two vertices, one on the upper base and one on the lower base, for each prism from the sequence $P_1, P_2, \ldots, P_n, \ldots$. Let $\mu(Q_1), \mu(Q_2), \ldots, \mu(Q_n), \ldots$ be the sequence of the corresponding surface areas.

Since $P_i \subset P_{i+1} \subset Q_i, \forall i \in N$ clearly the sequence $\{\mu(P_i)\}_{i=1}^{\infty}$ is nondecreasing and bounded from above, therefore convergent. Let $S$ be its limit: $\lim_{n \to \infty} \mu(P_n) = S$.

Let $a_n$ be the length of the apothem of the $P_n$ base. But the base of $P_n$ and the base of $Q_n$ are similar polygons with similarity coefficient $\frac{R}{a_n}$ and therefore $\mu(Q_n) = \mu(P_n) \cdot \frac{R}{a_n}$ and $\lim_{n \to \infty} \mu(Q_n) = S$, because of $\lim_{n \to \infty} a_n \cdot R = R$.

Suppose that there exists a number $\Omega$, bigger than the surface areas of all inscribed right prisms and smaller than the surface areas of all circumscribed right prisms. Since $\mu(P_n) < \Omega$ it follows $S \leq \Omega$. Analogously $\mu(Q_n) > \Omega$ follows $S \geq \Omega$, and therefore $S = \Omega$.

It remains to show that if $P$ and $Q$ are any two right prisms (inscribed and circumscribed) for their surface areas $\mu(P)$ and $\mu(Q)$ we have $\mu(P) < S < \mu(Q)$.

From the fact that $\mu(Q_n) > \mu(P)$ for all $n$ we conclude that $\mu(P) \leq S$ (true for each inscribed right prism). Equality is impossible because by doubling the faces of $P$ we obtain a
prism $P'$ that will have strictly bigger surface area (by Lemma), contradicting $\mu(P') \leq S$. We conclude that $\mu(P) < S$. Analogously $\mu(Q) > S$. This proves the theorem.

This Theorem motivates the following

**Definition 1:** For each right circular cylinder, the number, greater than the surface areas of the inscribed right prisms and smaller than the surface areas of the circumscribed right prisms is called surface area of the cylinder.

**Theorem 2:** Let $F_1,F_2,\ldots,F_n,\ldots$ be a sequence of right prisms, inscribed in a right circular cylinder, each containing the axis of the cylinder $OO'$ and the lengths of the biggest base edges of which go to zero. Then the sequence of the surface areas of the prisms converges to the surface area $S$ of the cylinder.

**Proof:** Since $F_n$ is inscribed, from Theorem 1 follows $\mu(F_n) < S$. Let $b_n$ be the length of the longest edge of the base $B_n$ of the prism $F_n$ and let $a_n$ be the distance from $O$ to this edge. Homothety in the plane of $k$ with center $O$ and coefficient $\frac{R}{a_n}$ transforms $B_n$ into a polygon $B_n'$, containing $k(O,R)$. The right prism $F_n'$ with the same height as the height of the cylinder and base $B_n'$ encompasses the cylinder.

Construct the polygon $B_n''$ having sides parallel to those of $B_n'$ and tangent to $k(O,R)$. Let $F_n''$ be the prism with base $B_n''$ and the same height as the height of the cylinder. Clearly $F_n''$ is encompassed by in $F_n'$ and therefore we have $\mu(F_n') \geq \mu(F_n'') > S$ from Lemma. Thus

$$\frac{\mu(F_n'')}{\mu(F_n')} \leq \frac{\mu(F_n'')}{\mu(F_n)} \leq \frac{R}{a_n}$$

and

$$S > \mu(F_n) > S \cdot \frac{a_n}{R}.$$  

From the condition $\lim_{n \to \infty} b_n = 0$ follows $\lim_{n \to \infty} a_n = R$ and after taking limit in the inequalities $S > \mu(F_n) > S \cdot \frac{a_n}{R}$ we obtain $\lim_{n \to \infty} \mu(F_n) = S$.

We compute $S$.

**Theorem 3:** The surface area of a right circular cylinder with radius of the base $R$ and height $h$ is $S = 2\pi Rh$.

**Proof:** Let $P_1,P_2,\ldots,P_n,\ldots$ be the sequence of Theorem 1. The sequence $p_1h,p_2h,\ldots,p_nh,\ldots$ of the surface areas of $P_1,P_2,\ldots,P_n,\ldots$, where $p_n$ are the corresponding perimeters, is convergent by Theorem 2. But then $S = \lim_{n \to \infty} p_nh = 2\pi R$, because $\lim_{n \to \infty} p_nh = 2\pi R$.

3. **Surface area of a right circular cone.**

The analogy between cones and cylinders allows us to just sketch the main points when introducing surface area of a cone.

**Theorem 4:** For each right circular cone there exists a unique number, greater than the surface areas of the inscribed right pyramids and smaller than the surface areas of the circumscribed right pyramids.

**Proof:** Take a right circular cone $K$ with base $k(O,R)$, vertex $V$ and a side edge with length $l$. We inscribe a regular triangular pyramid $P_1$, double its edges and the base and obtain $P_2$, double the edges of the base $P_2$, and obtain $P_3$ and so on. In each step the vertices of the base of one pyramid are among the vertices of the subsequent ones and all pyramids have vertex $V$. In this way we obtain a sequence $P_1,P_2,\ldots,P_n,\ldots$ of inscribed regular pyramids. Let $\mu(P_1),\mu(P_2),\ldots,\mu(P_n),\ldots$ be the sequence of their surface areas. We construct a sequence of circumscribed pyramids $Q_1,Q_2,\ldots,Q_n,\ldots$ by taking the tangent planes at each side edge containing $V$ and a vertex from the base, for each pyramid from the sequence $P_1,P_2,\ldots,P_n,\ldots$. Let $\mu(Q_1),\mu(Q_2),\ldots,\mu(Q_n),\ldots$ be the sequence of the corresponding surface areas.

Since $P_i \subset P_{i+1} \subset Q_i$, $\forall n \in N$ clearly the sequence $\{\mu(P_n)\}_{n=1}^{\infty}$ is nondecreasing and bounded from above, therefore convergent. Let $S$ be its limit: $\lim_{n \to \infty} \mu(P_n) = S$.

Let $a_n$ be the length of the base edge of $P_n$, $k$ be the length of the apothem of the base, $m_n$ be the length of the apothem of the pyramid and $b_n$ is the length of the base edge of
\( Q_n \). Then \( \frac{\mu(Q_n)}{\mu(P_n)} = \frac{b_n}{a_n} \frac{l}{m_n} \). But then
\[
\frac{b_n}{a_n} = \frac{R}{k_n}, \lim k_n = R \quad \text{and} \quad \lim m_n = l, \]
therefore \( \lim \mu(Q_n) = \lim \mu(P_n) = S \). The rest of the proof is similar to the proof of Theorem 1.

This Theorem motivates the following

**Definition 2:** For each right circular cone, the number, greater than the surface areas of the inscribed pyramids and smaller than the surface areas of the circumscribed pyramids is called surface area of the cone.

**Theorem 5:** Let \( F_1, F_2, \ldots, F_n, \ldots \) be a sequence of pyramids, inscribed in a right circular cone, each containing the axis of the cone \( OO' \) and the lengths of the biggest base edges of which go to zero. Then the sequence of the surface areas of the pyramids converges to the surface area \( S \) of the cone. Moreover any polyhedron, encompassing the cone has surface area bigger than \( S \).

**Proof:** Since \( F_n \) is inscribed, from Theorem 4 follows \( \mu(F_n) < S \). Let \( b_n \) be the length of the longest edge of the base \( B_n \) of the pyramid \( F_n \) and let \( a_n \) be the distance from \( O \) to this edge. Homothety in the plane of \( k(O, R) \) with center \( O \) and coefficient \( \frac{R}{a_n} \) transforms \( B_n \) into a polygon \( B_n' \), containing \( k(O, R) \). The pyramid \( F_n' \) with the same vertex \( V \) as the cone and base \( B_n' \) encompasses the cone. Construct the polygon \( B_n'' \) having sides parallel to those of \( B_n \) and tangent to \( k(O, R) \). Let \( F_n'' \) be the pyramid with base \( B_n'' \) and vertex \( V \). Clearly \( F_n'' \subset F_n' \) and therefore \( \mu(F_n') > \mu(F_n'') > S \) from Lemma. Moreover \( F_n' \) is encompassed by the pyramid, homothetic in space to the pyramid \( F_n \) with center of homothety \( O \) and coefficient \( \frac{R}{a_n} \), and therefore
\[
\frac{\mu(F_n''')}{\mu(F_n')} \leq \frac{\mu(F_n''')}{\mu(F_n')} \leq \left( \frac{R}{a_n} \right)^2. \]
This
\[
S > \mu(F_n) > S \left( \frac{a_n}{R} \right)^2. \]
From the condition
\[
limit_{n \to \infty} b_n = 0 \quad \text{follows} \quad \lim a_n = R \quad \text{and after taking limit in the inequalities}
\]
\[
S > \mu(F_n) > S \left( \frac{a_n}{R} \right)^2 \quad \text{we obtain}
\]
\[
\lim \mu(F_n) = S. \]

If \( G \) is a polyhedron encompassing the cone, it is clear that, because \( G \) encompasses \( F_n \) for each \( n \), we have \( \mu(F_n) < \mu(G) \). Taking limit we get \( S \leq \mu(G) \).

We compute \( S \).

**Theorem 6:** The surface area of a right circular cone with radius of the base \( R \) and lateral height \( l \) is \( S = \pi Rl \).

**Proof:** Let \( P_1, P_2, \ldots, P_n, \ldots \) be the sequence of Theorem 4. Let \( l_1, l_2, \ldots, l_n, \ldots \) be the apothems of these pyramids and \( p_1, p_2, \ldots, p_n, \ldots \) be the semi-adiameters of the bases. The sequence \( p_1l_1, p_2l_2, \ldots, p_nl_n, \ldots \) of the surface areas of \( P_1, P_2, \ldots, P_n, \ldots \) is convergent by Theorem 5. But then \( S = \lim p_nl_n = \pi Rl \), because \( \lim p_n = \pi R \) and \( \lim l_n = l \).

**Corollary:** Let \( P \) be a polyhedron, encompassed by a truncated cone \( K \) and \( Q \) be a polyhedron, encompassing \( K \). Then for the surface areas of those we have \( \mu(P) < \mu(K) < \mu(Q) \).

**Proof:** Follows from Theorem 5, the Lemma and the fact that a truncated cone, as a set, is a difference of two untruncated cones.

**5. Surface area of the sphere.** Consider a sphere \( T \) with center \( O \) and radius \( R \). A convex polyhedron \( P \) we will call inscribed in \( T \) if all the vertices of \( P \) lie on the sphere. The convex polyhedron \( Q \) we call circumscribed around \( T \) if all faces of \( Q \) are tangent to \( T \).

If we draw some longitudinal and latitudinal lines on the sphere, the intersection points will be vertices of an inscribed convex polyhedron. The faces of this polyhedron will be triangles (around the poles) and the rest trapezoids. A polyhedron of this type we will call a
pyramidal body. In the case when the longitudinal planes split the equator plane into equal sectors we call it a regular pyramidal body. In analogous way we define circumscribed pyramidal bodies.

**Theorem 7:** For each sphere there exists a unique number, greater than the surface areas of the inscribed pyramidal bodies and smaller than the surface areas of the circumscribed pyramidal bodies.

**Proof:** First we construct a sequence \( \{P_n\} \) of regular inscribed pyramidal bodies with surface areas \( \mu(P_n) \) for which we construct the sequence of pyramidal bodies \( \{Q_n\} \), homothetic to \( \{P_n\} \) and encompassing the sphere, with surface areas \( \mu(Q_n) \), and for which the sequences \( \{ \mu(P_n) \} \) and \( \{ \mu(Q_n) \} \) have same limits.

Let \( \varepsilon > 0 \). Consider the homotheties in space \( \varphi_0 \) and \( \varphi_1 \) with center \( O \) and coefficient \( R + \varepsilon \) and \( R - \varepsilon \) respectively. Let \( \varphi_0(T) = T_1 \) and \( \varphi_1(T) = T_2 \). We take a grid on \( T_1 \) in an appropriate way by longitudinal planes of equal angles and latitudinal planes, such that the diameter of any face of the inscribed regular pyramidal body is less than \( \varepsilon \). Let the regular pyramidal body thus obtained be \( P' \). Let \( \varphi_0^{-1}(P') = P \) which is a regular pyramidal body, that is clearly inscribed in \( T \). Let \( \varphi_1^{-1}(P') = Q' \) which is a regular pyramidal body, encompassing \( T \). Then

\[
\frac{\mu(P)}{\mu(Q')} = \left( \frac{R - \varepsilon}{R + \varepsilon} \right)^2.
\]

Let \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \ldots \) be a monotonously decreasing sequence of positive numbers, going to zero. For each \( \varepsilon_n \), as in the previous paragraph, we define a regular inscribed pyramidal body \( P_n \) (and its corresponding body \( Q_n \)). Thus we obtain sequences \( \{P_n\} \) and \( \{Q_n\} \) satisfying

\[
\lim_{n \to \infty} \mu(P_n) = \lim_{n \to \infty} \mu(Q_n) = l
\]

for some \( l \).

We construct a sequence of regular circumscribed pyramidal bodies \( \{Q_n\} \) by taking appropriate grids on \( T \) and taking the planes, tangent to \( T \) at the centers of each «spherical face». The grids are chosen so that the diameter of each face of \( Q_n \) is less than \( \varepsilon_n \). Using homotheties we construct a corresponding sequence \( \{P'_n\} \) of bodies, encompassed by the sphere \( T \) and satisfying

\[
\lim_{n \to \infty} \mu(P'_n) = \lim_{n \to \infty} \mu(Q_n) = m
\]

for some \( m \).

We use the Lemma to derive that \( m = 1 \) : \( \mu(Q_n) \) encompasses \( P'_n \) for all \( n \), therefore \( l \geq m \). For each \( n \), therefore \( l \geq m \). Let \( \varepsilon_n \) be a corresponding body such that \( \mu(Q_n) \) encompasses \( P_k \) for all \( k \) and \( n \), therefore \( l \geq m \).

Denote this limit by \( S \):

\[
\lim_{n \to \infty} \mu(P_n) = \lim_{n \to \infty} \mu(Q_n) = S.
\]

The rest of the proof is similar to the end of the proof of Theorem 1.

**Definition 3:** For each sphere, the number, greater than the surface areas of the inscribed pyramidal bodies and smaller than the circumscribed pyramidal bodies is called surface area of the sphere.

**Theorem 8:** The surface area of sphere with radius \( R \) is \( S = 4\pi R^2 \).

**Proof:** We construct a sequence of inscribed regular pyramidal bodies \( P_1, P_2, \ldots, P_n, \ldots \) for a sphere \( T \) of radius \( R \) in the following way: For each \( n \) we draw \( n \) equally spaced meridians and \( 2n \) latitudinal lines that split the “north” and “south” part into \( n \) sectors, such that the angles \( \angle A_i O A_{i+1} \) between two consecutive latitudinal planes, intersecting a meridian, are all equal to \( \frac{\pi}{2n} \) (see figure).

In each “truncated sphere” sector we inscribe a truncated cone \( K_{nm} \) (the north- and south- pole caps will be untruncated cones). We denote \( K_n = \bigcup K_{nm} \) – an inscribed conical body. For every \( \varepsilon > 0 \) from Theorem 7 we can construct a circumscribed pyramidal body \( Q \) for \( T \) such that

\[
\mu(Q) > \mu(T) > \mu(Q) - \varepsilon.
\]

Let \( Q_n \) be the part of the polyhedron \( Q \) situated between the two latitudinal planes of \( K_{nm} \). From the Corollary and the Lemma we have \( \mu(K_{nm}) < \mu(Q_n) \).

Taking sum and union we conclude

\[
\mu(P_n) < \mu(K_n) < \mu(Q_n).
\]

It follows

\[
\mu(K_n) \leq \mu(T) + \varepsilon
\]

and since \( \varepsilon > 0 \) was arbi-
trary we obtain \( \mu(K_n) \leq \mu(T) \) for all \( n \). We conclude \( \mu(P_i) < \mu(K_n) \leq \mu(T) \) and taking limit it follows \( \lim_{n \to \infty} \mu(K_n) = \mu(T) = S \) by the proof of Theorem 7.

We consider the northern hemisphere \( T' \).

\[
S = S_1 + S_2 + \ldots + S_{n-1} + S_n, \quad \text{where} \quad S_i \quad \text{is the surface area of the truncated cone defined by the points} \quad A_{i-1}, B_{i-1}, B_i, A_i \quad \text{and} \quad S_n \quad \text{is the cone defined by the points} \quad A_{n-1}, B_{n-1}, C \quad \text{(see figure)}. \]

To compute \( S_i \), we consider similar triangles \( A_{i-1}A_iD \), where \( D \) is the foot of the height of \( A_i \) to \( A_{i-1}B_{i-1} \) and triangle \( OM,E \), where \( E \) is the midpoint of the line segment \( P_{i-1}P_i \). But then

\[
S_i = \pi A_{i-1}A_i \frac{A_{i-1}B_{i-1} + A_iB_i}{2} = 2 \pi M_iO . P_{i-1}P_i .
\]

Also \( S_n = 2 \pi M_nO . P_{n-1}C \) and

\[
\frac{S}{2} = 2 \pi (M_1O . OP_1 + M_2O . P_1P_2 + \ldots + M_{n-1}O . P_{n-2}P_{n-1} + M_nO . P_{n-1}C),
\]

\( M_iO = R \cos \left( \frac{\pi}{4n} \right) \). Then

\[
\frac{S}{2} = 2 \pi R \cos \left( \frac{\pi}{4n} \right) (OP_1 + P_1P_2 + \ldots + P_{n-1}C) = 2 \pi R^2 \cos \left( \frac{\pi}{4n} \right).
\]

Taking limit we get \( S = 4 \pi R^2 \). This proofs the Theorem.

**Remark:** The convex polyhedron method that we used can be employed for more general convex surfaces (as done by Hadamard [1]). However from methodological point of view it has some advantage to consider “nicer” polyhedrons for defining the surface areas of rotational surfaces.

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